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Bosonic Super-Liouville System: Lax Pair and Solution

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We study the bosonic super-Liouville system, which is a statistical transformation of the super-Liouville system. The Lax pair for the bosonic super-Liouville system is constructed using the prolongation method, ensuring Lax integrability, and the solution to the equations of motion is also considered via Leznov-Saveliev analysis.

1. INTRODUCTION

Liouville and super-Liouville equations are important in a vast range of physical problems. For example, the Liouville equation is closely connected to string theory and two-dimensional gravity in the conformal gauge and is a very popular example of two-dimensional integrable field theory with conformal invariance, and the super-Liouville equation plays the same roles in super analogs of the above problems.

From the point of view of Toda lattice field theory, the Liouville equation is nothing but the simplest Toda field theory with the Toda lattice denoted by a single node—the Dynkin diagram of the Lie algebra $sl(2)$ [and the super-Liouville equation, which gauges the basic Lie superalgebra $osp(1|2)$ (Toppan, 1991), is the simplest one from the family of super Toda field theories]. It is remarkable that for each underlying Lie algebra $\mathscr{G}$ one can construct a Toda field theory. Analogously, for each basic Lie superalgebra one can construct a super Toda field theory. A more interesting fact is that for each Lie algebra $\mathscr{G}$ of rank $r > 1$, there exists a so-called bosonic super Toda theory (Chao, 1993; Chao and Hou, 1993, 1994; Hou and Chao, 1993),

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a kind of lattice field theory which can be viewed as the usual Toda field theory coupled to some bosonic "matter" fields, whose equations of motion look very similar to the super Toda equations written in component form for except the following two points[4]: (i) the Cartan matrices entering the equations of motion are different for bosonic super and true super Toda theories, since the underlying algebras are different; (ii) the bosonic super Toda theory contains only bosonic fields and hence does not yield a true supersymmetry. However, despite these differences, the bosonic super Toda theory does yield very nice mathematical properties both as integrable and conformal field-theoretic models; in particular, such a model is intimately related to the $W_n^{(2)}$ algebra if the underlying gauge group is chosen to be $SL(n, R)$ (Chao and Hou, 1994). Moreover, it was recently argued in Ferreria *et al.* (1995) and Gervais and Saveliev (1995) that although classically the extended Toda theories such as the bosonic super-Liouville theory contain only bosonic fields, their quantum versions might give rise to some fermionic degrees of freedom and may have relevant applications in photoelectronic problems. Due to both their mathematical beauty and the potential physical significance, much effort has been given to the study of bosonic super Toda theories (Chao, 1993, 1994, 1995; Chao and Hou, 1993, 1994; Hou and Chao, 1993).

Two puzzling points are worth further effort. First, since the equations of motion for bosonic super Toda and true super Toda theories are so much alike, one naturally expects to establish some relationship between these two kinds of theories; second, though the bosonic super Toda theory exists for almost all underlying Lie algebras, the simplest rank-one Lie algebra $sl(2)$ is excluded from this picture and therefore no bosonic super-Liouville model exists along the above line.

A naive answer to the first problem might be that the bosonic super Toda theories are just the statistically transformed super Toda theories, i.e., by replacing all the fermionic fields in super Toda theories by bosonic ones, one gets a bosonic super Toda theory. But this cannot be true, as the Cartan matrices entering the equations of motion are quite different for both kinds of theories. However, this naive idea might be a useful clue for constructing a bosonic super-Liouville model, and in this paper we adopt such a technique to define a bosonic super-Liouville model.

We start from the supersymmetric Liouville equation

$$D_+D_-\Phi = \exp(\Phi) \tag{1}$$

[4]Compare the equations of motion for bosonic super Toda theories in Chao (1993), Chao and Hou (1993, 1994), Hou and Chao (1993) and that of supersymmetric Toda theories in, e.g., Au and Spence (1995).

where we have chosen

$$D_\pm = \frac{1}{3}\frac{\partial}{\partial\theta_\mp} \pm \theta_\mp \frac{\partial}{\partial_{x_\pm}}$$

and

$$\Phi = \phi + 3\sqrt{2}\,(\theta_+\psi_- + \theta_-\psi_+) + 6\theta_+\theta_- F$$

so that, in component form, equation (1) can be rewritten as follows:

$$\begin{aligned} \partial_+\partial_-\phi &= 18\psi_+\psi_- e^{\phi} + 4e^{2\phi} \\ \partial_+\phi_- &= 3\psi_+ e^{\phi} \\ \partial_-\psi_+ &= 3\psi_- e^{\phi} \end{aligned} \tag{2}$$

These equations have exactly the same form as an extended Liouville equation obtained by Chao (1993, 1994) except that the fields $\psi_\pm$ in (2) are fermionic. We call equation (2) with $\psi_\pm$ changed into bosonic fields by statistically transmitted super-Liouville equation or bosonic super-Liouville equation (BSLE), and the present paper is devoted to the study of the integrability of that equation. We stress that in the BSLE no signature change occurs in front of the $\psi_+\psi_-$ term, when the order of ψ_+ and ψ_- is reversed

Before going into detailed studies, we mention that equation (2), viewed as a BSLE, represents the usual Liouville system coupled to a pair of external fields $\psi_\pm$. Moreover, these external fields do not possess mass, because the whole system of equations of motion is conformally invariant, i.e., the coordinate system (x_+, x_-) undergoes the conformal transformation

$$x_\pm \rightarrow f_\pm(x_\pm)$$

the equations of motion will be left invariant provided the fields ϕ, $\psi_\pm$ transform as

$$\begin{cases} \phi \rightarrow + \ln(f'_+)^{1/2}\,((f'_-)^{1/2} \\ \psi_\pm \rightarrow (f'_\pm)^{1/2}\,\psi_\pm \end{cases}$$

It is interesting to see that the statistical transformation from the super-Liouville equation to the BSLE also changes the fields ψ_+ and ψ_- from the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ of the Lorentz group to that of the classical conformal group.

2. LAX-PAIR AND SYMMETRY ALGEBRAS FOR BSLE

In this section we address the problem of integrability of the BSLE (2). A system of nonlinear partial differential equations is said to be integrable if it is a Hamiltonian system and possesses an infinite number of Poisson-commuting integrals of motion. This is the classical Liouville sense of integrability. Another, slightly weaker definition of integrability identifies the system with the compatibility condition of a system of linear auxiliary problems, i.e., the Lax pair. Lax integrability will be identical to Liouville integrability if the Lax system admits a fundamental Poisson structure and this Poisson structure can be recast into the form of a classical Yang–Baxter formalism. Therefore the first step to considering the integrability of the BSLE either in the Liouville sense or in the Lax sense is to find its Lax formalism, and to this end the prolongation approach (Walquist and Estabrook, 1975; Lu and Li, 1989a,b) is preferred.

To begin with, we introduce a transformation of independent variables

$$x_+ \to \frac{x_0 + x_1}{2}, \qquad x_- \to \frac{x_0 - x_1}{2}$$

which leads to the changes $\partial_\pm \to \partial_0 \pm \partial_1$ of the derivatives.

Setting $\pi_0 = \partial_0\phi$, $\pi_1 = -\partial_1\phi$, we can express the system (2) by the following set of rank-two differential forms on the space of variables (x_0, x_1, ϕ, ψ_+, ψ_-, π_0, π_1):

$$\begin{aligned}
\alpha_1 &= d\psi_+ \wedge dx_1 - dx_0 \wedge d\psi + - 3\psi_- e^{\phi}\, dx_0 \wedge dx_1 \\
\alpha_2 &= d\psi_- \wedge dx_1 + dx_0 \wedge d\psi_- - 3\psi_+ e^{\phi}\, dx_0 \wedge dx_1 \\
\alpha_3 &= d\phi \wedge dx_1 - \pi_0\, dx_0 \wedge dx_1 \\
\alpha_4 &= d\phi \wedge dx_0 - \pi_1\, dx_0 \wedge dx_1 \\
\alpha_5 &= d\pi_0 \wedge dx_1 - d\pi_1 \wedge dx_0 - (18\psi_+\psi_- e^{\phi} + 4e^{2\phi})\, dx_0 \wedge dx_1
\end{aligned} \tag{3}$$

On the intersection with the space of independent variables (x_0, x_1) the system (2) will be reproduced. It is easy to check that the system (3) of two-forms generates a closed ideal in the sense that

$$d\alpha_i = \eta_{ij}\alpha_j$$

for some one-forms η_{ij}. Given the system (3), we now assume that the (enlarged) prolongation (Lu and Li, 1989a,b) form is

$$\omega = -dT + F(\phi, \psi_+, \psi_-, \pi_0, \pi_1)\, T\, dx_0 + G(\phi, \psi_+, \psi_-, \pi_0, \pi_1) T\, dx_1 \tag{4}$$

where F and G are functions of the indicated variables taking values in some undetermined Lie algebra, and the newly introduced "pseudopotential" T lies in the group generated by that Lie algebra.

From the integrability condition

$$d\omega \in I(\omega, \alpha)$$

where $I(\omega, \alpha)$ is an ideal generated by the set $\{\alpha_i\}$ and $\{\omega\}$, we have the following equations for F and G:

$$
\begin{aligned}
F_{\psi_+} - G_{\psi_+} &= 0 \\
F_{\psi_-} + G_{\psi_-} &= 0 \\
F_{\pi_0} - G_{\pi_1} &= 0 \\
F_{\pi_1} + G_{\pi_0} &= 0 \\
[F, G] + \pi_1 F_\phi + \pi_0 G_\phi + 3e^{\phi}(\psi_- F_{\psi_+} &+ \psi_+ G_{\phi_-}) \\
-(18\psi_+\psi_- e^{\phi} + 4e^{20})F_{\pi_1} &= 0
\end{aligned}
\tag{5}
$$

where $[F, \mathrm{G}] = FG - GF$. Solving the system of equations (5), we get

$$F = -\frac{1}{2}[\pi_1 L_0 + 3\psi_+ e^{\frac{\phi}{2}} L_1 - 3\psi - e^{\frac{\phi}{2}} L_{-1} + e^{\phi} L_2 - e^{\psi} L_{-2}]$$

$$G = \frac{1}{2}[\pi_0 L_0 + 3\psi_+ e^{\frac{\phi}{2}} L_1 + 3\psi_- e^{\frac{\phi}{2}} L_{-1} + e^{\phi} L_2 + e^{\phi} L_{-2}]$$

where $L_i = i = 0, \pm 1, \pm 2$, are operators satisfying the following commutation relations:

$$
\begin{aligned}
[L_0, L_1] &= -L_1, & [L_0, L_{-1}] &= L_{-1} \\
[L_0, L_2] &= -2L_2, & [L_0, L_{-2}] &= 2L_{-2} \\
[L_1, L_{-1}] &= -2L_0, & [L_1, L_{-2}] &= 3L_{-1} \\
[L_{-1}, L_2] &= -3L_1, & [L_2, L_{-2}] &= 4L_0
\end{aligned}
\tag{6}
$$

Notice that the system (6) does not yet generate a closed algebra. However, one can easily see that all relations in (6) can be rewritten in a unified form

$$[L_n, L_m] = (n - m)L_{n+m} \tag{7}$$

for $n, m = 0, \pm 1, \pm 2$. Defining new generators iteratively by

$$L_{m+2} = \frac{1}{m}[L_{m+1}\ L_1], \qquad L_{m-2} = \frac{1}{m}[L_{-1}, L_{-m-1}], \qquad m \geq 1$$

we find that equation (7) will close over the generators L_j, $j = 0, \pm 1, \pm 2, \ldots$. This is the well-known Witt algebra or "centerless Virasoro algebra."

Now intersecting the prolongation form (4) on the solution manifold (x_+, x_-), we obtain the Lax pair for the BSLE (2),

$$\partial_+ T = (F + G)T$$
$$\partial_- T = (F - G)T \tag{8}$$

The existence of the Lax pair (8) ensures that the BSLE (2) is integrable in the Lax sense. However, since no Hamiltonian structure is currently known for the BSLE, the Liouville integrability cannot be established at this point.

Notice that the Lax pair (8) involves the generator of the Witt algebra with degrees ranging from -2 to 2. It is well known that the Witt algebra does not contain any finite-dimensional subalgebra of dimension greater than 3. Therefore the Witt algebra is the only possible gauge algebra of the Lax system (8). Moreover, as there is no nondegenerate symmetric bilinear form on the Witt algebra, it is hard to obtain a Lagrangian formulation for the BSLE as in the conventional Toda case by taking the trace of A_+A_-, with $A_\pm$ being the Lax potentials. Actually, if the Lagrangian is indeed in the form of a trace over A_+A_- in the case of the BSLE, then this would lead to the conclusion that the BSLE is a topological theory because the Lagrangian is identically zero. Whether this is true or not deserves further study.

3. SOLUTION OF BSLE

Given the Lax pair (8), we can now consider the possible solutions of the BSLE (2) using the Leznov–Saveliev analysis.

For convenience we chose the following specific gauges for the Lax pair of the BSLE:

$$\partial_+ T_L = (\partial_+\phi L_0 + 3\psi_+ L_1 + L_2)T_L$$
$$\partial_- T_L = -(3\psi_- e^{\phi} L_{-1} + e^{2\phi} L_{-2})T_L \tag{9}$$

and

$$\partial_+ T_R = (3\psi_+ e^{\phi} L_1 + e^{2\phi} L_2)T_R$$
$$\partial_- T_R = -(\partial_-\phi L_0 + 3\psi_- L_{-1} + L_{-2})T_R \tag{10}$$

where

$$T_L = e^{\phi L_0/2}\, T, \qquad T_R = e^{-\phi L_0/2}\, T \tag{11}$$

Now let us choose some highest weight representation of the Witt algebra with highest weight h and denote the highest weight vector by $|h\rangle$. The dual of $|h\rangle$ is denoted $\langle h|$. The highest weight conditions read

$$
\begin{aligned}
L_0|h\rangle &= h|h\rangle, \qquad \langle h|L_0 = \langle h|h \\
L_n|h\rangle &= 0, \qquad \langle h|L_{-n} = 0 \qquad (n > 0) \\
\langle h|h\rangle &= 1
\end{aligned}
\tag{12}
$$

From (9), (10), and (12), it follows that

$$
\langle h|\partial_- T_L = 0, \qquad \partial_+ T_R^{-1}|h\rangle = 0 \tag{13}
$$

and hence the vectors

$$
\xi(x_+) = \langle h|T_L, \qquad \bar{\xi}(x_-) = T_R^{-1}|h\rangle \tag{14}
$$

are chiral, namely

$$
\partial_- \xi(x_+) = 0, \qquad \partial_+ \bar{\xi}(x_-) = 0 \tag{15}
$$

Moreover, defining

$$
\begin{aligned}
\bar{T}_L &= e^{\psi_+ L_{-1}}\, T_L \\
\bar{T}_R &= e^{-\psi_- L_1}\, T_R
\end{aligned}
$$

we find from an easy calculation that

$$
\langle h|L_1 \partial_- \bar{T}_L = 0, \qquad \partial_+ \bar{T}_R^{-1} L_{-1}|h\rangle = 0 \tag{16}
$$

showing that the vectors

$$
\zeta(x_+) = \langle h|L_1 \bar{T}_L, \qquad \bar{\zeta}(x_-) = \bar{T}_R^{-1} L_{-1}|h\rangle \tag{17}
$$

are also chiral

$$
\partial_- \zeta(x_+) = 0, \qquad \partial_+ \bar{\zeta}(x_-) = 0 \tag{18}
$$

From equations (13–18), a straightforward calculation gives

$$
\begin{aligned}
\xi(x_+)\bar{\xi}(x_-) &= e^{h\phi} \\
\zeta(x_+)\bar{\xi}(x_-) &= 2h\psi_+ e^{h\phi} \\
\xi(x_+)\bar{\zeta}(x_-) &= 2h\psi_- e^{h\phi}
\end{aligned}
$$

which in turn gives a formal solution to the BSLE

$$\phi = \frac{1}{h}\ln[\xi(x_+)\bar{\xi}(x_-)] \tag{19}$$

$$\psi_+ = \frac{1}{2h}\frac{\zeta(x_+)\bar{\xi}(x_-)}{\xi(x_+)\bar{\xi}(x_-)} \tag{20}$$

$$\psi_- = \frac{1}{2h}\frac{\xi(x_+)\bar{\zeta}(x_-)}{\xi(x_+)\bar{\xi}(x_-)} \tag{21}$$

Some remarks are in order. First, one could be quite dubious of the correctness of the assumption of the highest weight conditions (12). Indeed, it is known from the study of conformal field theory that no nontrivial *unitary* highest weight representations exists for Virasoro algebra at the center $c = 0$. However, as we are using the Witt algebra as a gauge algebra of our Lax system, we are not concerned with the unitarity of the representation and so are free to choose the nonunitary representations in (12). Actually, the choice of nonunitary representations in (12) is not unavoidable if we introduce an extra auxiliary field, say ρ, and modify the Lax system (8) to the form

$$\begin{aligned} \partial_+ T &= (\partial_+\rho c + F + G)T \\ \partial_- T &= (-\partial_-\rho c + F - G)T \end{aligned} \tag{22}$$

and change the gauge algebra (6) into the full Virasoro algebra

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}$$

One can show that such modifications do not change the equations of motion for ϕ, $\psi_\pm$ and only give rise to a new equation for the auxiliary field ρ,

$$\partial_+\partial_-\rho + 2\exp(2\phi) = 0$$

The modified Lax system (22) can then be treated in exactly the same way as above and one can choose unitary highest weight representations of the Virasoro algebra in place of the nonunitary representations in (12).

Another remark is as follows. Though the solution of the BSLE (2) can be expressed in the form of (21), the chiral vectors cannot be regarded as arbitrary, because they are defined from the nonchiral objects T_L, T_R and $\bar{T}_L$, $\bar{T}_R$ subject to nontrivial constraints (the Lax pair). The explicit solution of the BSLE therefore cannot be obtained in this way. In the conventional Liouville and Toda cases, one can, however, make a similar construction starting not from the specific gauges (9) and (10) of the Lax pair, but from the set of so-called Drinfeld–Sokolov systems. In the present case such systems would look like

$$\partial_+ Q = (\partial_+ k(x_+)L_0 + 3p(x_+)L_1 + L_2)Q, \qquad \partial_- Q = 0$$
$$\partial_- \bar{Q} = \bar{Q}(\partial_- k(x_-)L_0 + 3p(x_-)L_{-1} + L_{-2}, \qquad \partial_+ \bar{Q} = 0$$

with some *arbitrary* chiral functions $k(x_\pm)$ and $p(x_\pm)$. Unfortunately, we have been unable to obtain exact solutions to (2) using the above Drinfeld–Sokolov systems.

4. DISCUSSION

In this paper, we have identified the Lax integrability for the BSLE (2) by using an enlarged prolongation approach. The same Lax pair will be obtained if we use the original scalar form of the prolongation forms as in the classic paper of Walquist and Estabrook (1975).

On the other hand, we expressed the solution of the BSLE in terms of some chiral vectors obtained from the action of the Lax system in some specific gauges which is in complete analogy to the conventional Toda and bosonic Toda cases. However, as mentioned at the end of the last section, the Drinfeld–Sokolov construction of solutions for the BSLE is not established and this may be one of the subtle points where the BSLE behaves in a different way from the bosonic super Toda theories.

To conclude this paper, we briefly point out some related open problems:

1. It's easy to see that the quantity $\frac{1}{2}\partial_+\phi\partial_-\phi + 18\psi_+\psi_- e^{\phi} + 2e^{2\phi}$ is a Lagrangian for the "Liouville part" ϕ in the BSLE (2). However no Lagrangian expression for $\psi_\pm$ is known. A principal reason is that $\psi_\pm$ are chiral fields of first order, and it seems interesting to see whether one can construct a Hamiltonian or Lagrangian formalism for the whole BSLE via the Dirac method.

2. It was shown in the introduction that the BSLE is conformal invariant and thus admits a $Witt_L \otimes Witt_R$ symmetry algebra. Is there any relationship between the conformal symmetry algebra and the gauged Witt algebra?

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